Puzzle of the Week

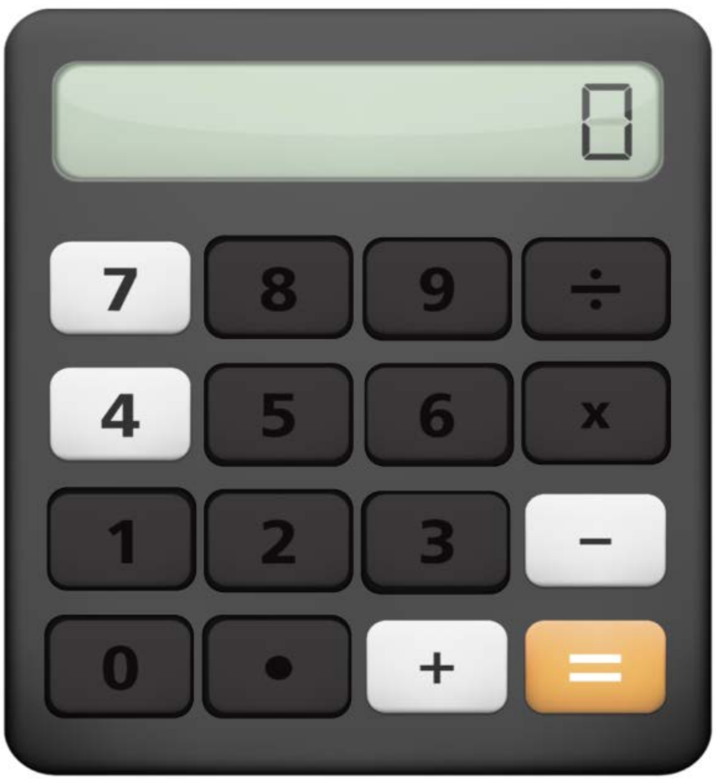
Broken Calculator – 1

You have a calculator that is badly broken. Its only working keys are 4, 7, +, and -. Even with this limited ability, it is possible to make every number. For example:

$$1 = 4 + 4 - 7$$

$$2 = 4 + 4 + 4 + 4 - 7 - 7$$

THE CHALLENGE: Show that this calculator can make all the numbers from 1 to 12.



EXPLORATION: Replace 4 and 7 by other pairs of numbers. When is it possible to get all the numbers from 1 to 12, and when is it impossible? Can you look at a pair of numbers and predict what will happen?

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Broken Calculator – 1 – Notes

THE CHALLENGE: The students are experimenting with all the numbers that can be produced by taking any multiple of 4 and adding it to any multiple of 7. Without being told this, they are looking at what is called Bezout's Theorem. That theorem says that all possible combinations of the multiples (both positive and negative) of two numbers is the set of multiples of the greatest common factor (divisor) of the two numbers. The greatest common factor of 4 and 7 is 1, so we should be able to get all possible numbers. Furthermore, there will be an infinite number of ways to get each one.

Here are a few samples for the numbers from 1 to 12:

- 1 = 4 + 4 - 7 = 9 x 4 - 5 x 7 = 16 x 4 - 9 x 7
- 2 = 7 + 7 - 4 - 4 - 4 = 6 x 7 – 10 x 4 = 10 x 7 - 17 x 4
- 3 = 7 - 4 = 5 x 7 - 8 x 4 = 9 x 7 - 15 x 4
- 4 = 4 = 4 x 7 - 6 x 4 = 8 x 7 - 13 x 4
- 5 = 3 x 4 - 7
- 6 = 2 x 7 - 2 x 4
- 7 = 7
- 8 = 2 x 4
- 9 = 3 x 7 - 3 x 4
- 10 = 2 x 7 - 4
- 11 = 4 + 7
- 12 = 3 x 4

Once you have a combination for the greatest common factor, you can use it to get any multiple of the greatest common factor. For 4 and 7, one way to get 1 is as 2 x 4 - 7. We can get any other number, such as 23, by writing 23 = 23 (2 x 4 - 7) = 46 x 4 - 23 x 7. Once we have one solution for a number, we can get all the other solutions by adding 0 to it (I know that sounds odd). For example, we have 1 = 2 x 4 - 7 = (2 x 4 - 7) + (7 x 4 - 4 x 7) = 9 x 4 - 5 x 7). We can add any multiple we like of (7 x 4 - 4 x 7) and it won't change the value.

EXPLORATION: As mentioned above, this process will produce exactly all the multiples of the greatest common factor of the two numbers. If that GCF is 1, then all numbers will be produced.

Take 4 and 6 for example. Their GCF is 2. Any combination of multiples of 4 and 6 will always be even, so it is impossible to produce every number. This becomes more apparent if you think of 4 and 6 as 2 times 2 and 3. You know that sums of multiples of 2 and 3 will produce all numbers because they are relatively prime. However, 4 and 6 will take each of those sums and multiply them by 2, so that's why you will get all multiples of 2!